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Real-function inequalities from conformal maps

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Abstract. We establish several new inequalities for real functions that are the restriction to the real axis of conformal maps of the upper half-plane into itself. Our various results involve the average derivatives, geometric mean derivatives, cross-ratios and Schwarzian derivatives of such functions. Connections with the hyperbolic metric and theory of monotone matrix funtions are mentioned. Applications to conformal field theory, two-dimensional phase transitions and special functions are made.

1. Introduction

In this paper we consider real functions that are restrictions to the real axis of conformal maps of the upper half-plane into itself. Several new inequalities involving average derivatives, geometric mean derivatives, cross-ratios and Schwarzian derivatives of such functions are presented. Our method of derivation, based on harmonic function theory, may be understood via simple electrostatic arguments. Alternate derivations of some of our results, making use of the hyperbolic distance or the theory of monotone matrix functions, are mentioned. A few applications to conformal field theory for phase transitions in two dimensions and special functions are made. We also point out a connection with previous research on scattering theory.

2. Derivations

In this section we derive the inequalities. Our method is a straightforward application of harmonic function theory. The basis of our results is the fact that at a given point, the potential of a positive point charge in a two-dimensional region with vanishing boundary conditions must decrease if the region is shrunk.

We also mention alternate derivations for some of the inequalities from hyperbolic geometry or the theory of monotone matrix functions. Thus, in what follows, theorems 1 and 2 are already known. They are included here to lay the groundwork for theorem 3 and its consequences. These, to our knowledge, are new.

Let D denote the upper half of the z-plane and let $z_1, z_2 \in D$. Consider the function

$$\psi(z_1) = -\ln\left(\frac{z_1 - z_2}{z_1 - z_2^*}\right) \qquad (1)$$

where the asterisk denotes complex conjugation. The real part of this,

$$\Psi(x_1, y_1) = -\ln\left|\frac{z_1 - z_2}{z_1 - z_2^*}\right| \tag{2}$$

where $z_1 = x_1 + \mathrm{i}y_1$, is interpretable as the electrostatic potential (in two dimensions) of a positive point charge of strength 2π located at $z_2 = x_2 + \mathrm{i}y_2$ with $\Psi = 0$ on the boundary Γ of D (i.e. the entire real axis including the point $z_1 = \infty$).

Now let $w = w(z)$ be any conformal mapping of D into a subset $w(D)$, $D \supseteq w(D)$ and consider the function

$$\phi(w) \equiv \psi(z(w)). \tag{3}$$

Then

$$\Phi(u_1, v_1) = \mathrm{Re}\{\phi(w_1)\} = \Psi(x_1, y_1) \tag{4}$$

where $w_1 = u_1 + \mathrm{i}v_1$, is the potential of a positive point charge of strength 2π located at $w_2 = u_2 + \mathrm{i}v_2$ with $\Phi = 0$ on the boundary $w(\Gamma)$ of $w(D)$. By hypothesis, this boundary can include part of the real axis but must not extend below it.

Note that here, and at many points below, we are considering the points z_1, z_2 as *dependent* variables, fixed via the *inverse* of the conformal map w by the location of the charge w_2 and field view w_1 in the w plane. The function Ψ appears in this context in an auxiliary role, via (4), in expressing the potential Φ in the region $w(D)$.

Finally, for comparison we consider the function

$$\theta(w_3) = -\ln\left(\frac{w_3 - w_2}{w_3 - w_2^*}\right) \tag{5}$$

with $w_3 \in D$. Its real part,

$$\Theta(u_3, v_3) = -\ln\left|\frac{w_3 - w_2}{w_3 - w_2^*}\right| \tag{6}$$

where $w_3 = u_3 + \mathrm{i}v_3$, is the potential of a positive point charge of strength 2π located at $w_2 = u_2 + \mathrm{i}v_2$ with $\Theta = 0$ on the entire real axis (including $w_3 = \infty$).

Now since the vanishing boundary condition for Φ defines a region enclosed by the corresponding region for Θ, at a given point $\Theta \geqslant \Phi$. This is exactly the content of:

Theorem 1.

$$\left|\frac{z_1 - z_2}{z_1 - z_2^*}\right| \geqslant \left|\frac{w_1 - w_2}{w_1 - w_2^*}\right|. \tag{7}$$

Proof. Let

$$\chi(w_1) \equiv \theta(w_1) - \phi(w_1). \tag{8}$$

Note that χ is defined only for $w_1 \in w(D)$. Its real part

$$\begin{aligned}\Xi(u_1, v_1) &= \mathrm{Re}\{\chi(w_1)\}\\ &= \Theta(u_1, v_1) - \Phi(u_1, v_1)\\ &= \Theta(u_1, v_1) - \Psi(x_1, y_1)\end{aligned} \tag{9}$$

is the potential in $w(D)$ due to zero charge in $W(D)$ with $\Xi=\Theta$ on the boundary $w(\Gamma)$ of the mapped $w(D)$. From (6) we see that $\Theta(w_1)\geqslant 0$ for every point w_1 on $w(\Gamma)$. Therefore Ξ, which can achieve its minimum only on the boundary $w(\Gamma)$ of $w(D)$, must be $\geqslant 0$ for all $w_1\in w(D)$. The desired result (7) then follows from equations (2), (4), (6) and the monotonicity of the ln function. As mentioned, this theorem can be understood as the statement that, with vanishing boundary conditions, the potential of a positive charge in the upper half-plane decreases if we shrink the half-plane.

Remark. Theorem 1 may also be established by consideration of the hyperbolic distance

$$\rho(z_1, z_2)=\ln\left(\frac{|z_1-z_2^*|+|z_1-z_2|}{|z_1-z_2^*|-|z_1-z_2|}\right).$$

In the upper half-plane this is the distance between two points measured with the hyperbolic or Poincaré metric $ds_h^2=ds^2/y^2$ [1]. For a map of the unit circle into itself, the hyperbolic distance cannot increase [2]. By a coordinate transformation this result also holds in the half-plane so that $\rho(w(z_1), w(z_2))\leqslant\rho(z_1, z_2)$. Equation (7) then follows immediately since the quantitites compared are monotonic functions of ρ, i.e.

$$\left|\frac{z_1-z_2}{z_1-z_2^*}\right|=\tanh\left(\frac{\rho(z_1, z_2)}{2}\right).$$

The proof that ρ cannot increase follows from (and is in fact equivalent to [3]) the Schwarz–Pick lemma. The Schwarz lemma, in turn, is a consequence of the maximum principle for analytic functions, which is cental in our proof of theorem 1.

Next, specialize to conformal mappings w which take at least two points (x_1, x_2) of the real z axis into the points (u_1, u_2) of the real w axis. Then we are in a position to prove:

Theorem 2. (Geometric mean value inequality):

$$\left|\frac{u_1-u_2}{x_1-x_2}\right|\leqslant(u_1'u_2')^{1/2} \qquad (10)$$

where the prime denotes differentiation. Note that the conditions of the theorem ensure that all quantities in (10) are real.

Proof. Consider equation (7) with $z_i=x_i+iy_i$, $i=1,2$. Let $y_i\to 0$, so that

$$\left|\frac{z_1-z_2}{z_1-z_2^*}\right|\to 1-2\frac{y_1y_2}{X^2} \qquad (11)$$

where $X=|x_1-x_2|$. Write $w_i=u_i+iv_i$, let $U=|u_1-u_2|$, and note that $v_i\to u_i'y_i$. Substituting (11) and its analogue for w-plane quantities into (7) leads immediately to the result (see figure 1). Note that the second term on the RHS of (11) is proportional to the interaction energy in the infinite plane of a dipole of moment y_1 at x_1 with one of moment y_2 at x_2. Transforming this arrangement to the w plane changes the separation of the dipoles and multiplies each moment by u', due to the rescaling of disances. Hence (10) may be understood as the statement that the conformal map increases the dipole interaction energy in the infinite plane.

Remark. Theorem 2 also follows from the classic work on monotone matrix functions

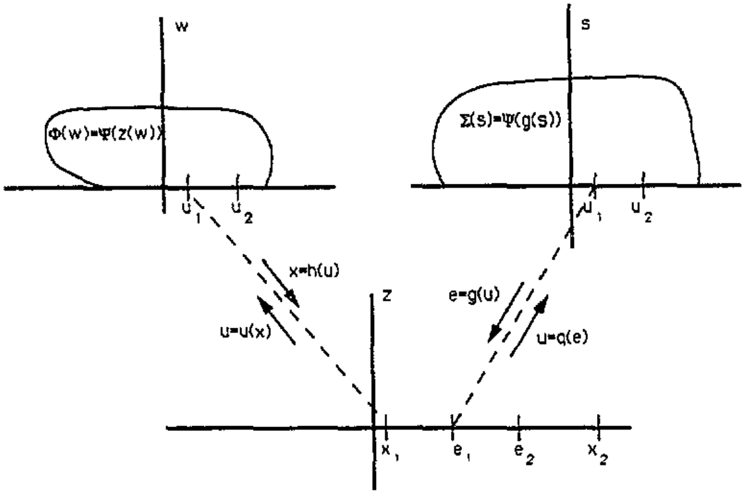

Figure 1. The geometry of theorems 1–3.

by Löwner [4] (for recent references see [5, 6]). For a function to be matrix monotone, it is necessary and sufficient that its analytic continuation maps the upper half-plane into itself. Another necessary and sufficient condition is that the kernel

$$K(x_1, x_2) = \frac{u_1 - u_2}{x_1 - x_2}, \qquad K(x, x) = u'(x)$$

constructed from such a function be positive semi-definite for points of the type considered. Hence the determinant of K is non-negative, and (10) follows immediately ([5], p 552).

Theorem 2 may be generalized as follows. Consider a second analytic function $s = s(z)$, $s = q + ir$, that maps D into a subset of the upper half-plane $D \supseteq s(D) \supseteq w(D)$. Assume the boundary $w(\Gamma)$ of $w(D)$ and the boundary $s(\Gamma)$ of $s(D)$ share at least two points. Take the common boundary points to be (u_1, u_2) on the real axis. Place a charge at w_2 and consider the potential at w_1 in each region, as above. Denote the potential in $s(D)$ by Σ. By simple adoption of the proof of theorem 1, $\Sigma \geqslant \Phi$ in $w(D)$, the smaller region. Hence

$$\left| \frac{z_1 - z_2}{z_1 - z_2^*} \right| \geqslant \left| \frac{g_1 - g_2}{g_1 - g_2^*} \right| \tag{12}$$

where g is the inverse of s so that $w_i = s(g_i)$. Now let $z_i = x_i + iy_i$, as above, and set $g_i = e_i + if_i$, so that (w_1, w_2) tend to the common boundary points (u_1, u_2) as $y_i \to 0$ (see figure 1). Expanding $w_i = s(e_i + if_i)$ and $w_i = w(x_i + iy_i)$ and comparing the results leads immediately to $f_i = (u_i'/q_i')y_i$, where $q_i' = q'(e_i)$. Comparing this with (12) results in

Theorem 3.

$$|e_1 - e_2|(q_1' q_2')^{1/2} \leqslant |x_1 - x_2|(u_1' u_2')^{1/2}. \tag{13}$$

The sense of the inequality in (13) reflects the fact that the map w contracts more than s. If we take $s = z$, theorem 3 reduces to theorem 2. We emphasize again that e_i and x_i are considered as functions of u_i in (13), via the relations $u_i = u(x_i) = q(e_i)$,

$i=1,2$. To our knowledge, theorem 3 and its consequences, such as theorem 4, are new.

The conditions on theorem 3 may be loosened somewhat. If the two common boundary points are not on the real axis, (13) holds for appropriately rotated maps. It is not necessary that $s(D)$ (or $w(D)$) be a subset of the half-plane, however, s must be such that $s(D)$ includes $w(D)$ and does not overlap itself. In general, some quantities in (13) may be complex, so that a rotation is necessary, in addition one can have (after rotation) $\mathrm{sign}(q_1') = -\mathrm{sign}(q_2')$ and similarly for u, so that a minus sign is needed inside the square root.

Next, assume that w and s share a finite region of common boundary points along the real axis. The region may be composed of disjoint pieces. Let h be the inverse of u, i.e. $x=h(u)$. Using this function and $e=g(q)$, the inverse of s, it is easy to re-express (13) as

$$\frac{g_1'g_2'}{(g_1-g_2)^2} \geqslant \frac{h_1'h_2'}{(h_1-h_2)^2}$$
$$\frac{\partial^2 \ln|g(u_1)-g(u_2)|}{\partial u_1 \partial u_2} \geqslant \frac{\partial^2 \ln|h(u_1)-h(u_2)|}{\partial u_1 \partial u_2} \tag{14}$$

where use has been made of the fact that $q(e_i)=u_i$. At this point it may be helpful to refer to figure 1, where the various functions ad variables are illustrated. Now consider, for any four values of an arbitrary function f, the cross-ratio

$$C(f) \equiv \frac{f_2-f_4}{f_2-f_3}\,\frac{f_1-f_3}{f_1-f_4}. \tag{15}$$

Successively integrating (14) between $u_1 \leqslant u_2$ and $u_3 \leqslant u_4$ then leads to, under the (strong) conditions of theorem 3,

Theorem 4. (Cross-ratio inequality):

$$C(g) \geqslant C(h). \tag{16}$$

Here, for the integrations over u to be possible, the limits of integration (u_1, u_2) and (u_3, u_4) must each belong to a single contiguous common boundary region. In deriving (16), we have taken advantage of the monotonicity of g or h in any such region, which follows directly from theorem 2. In addition, we have set $u_2 \leqslant u_3$. This and the monotonicity ensure that the cross-ratios are positive—otherwise (16) (as well as the next two inequalities) holds with each C replaced by its absolute value.

One may similarly integrate theorem 2 directly with the result

$$C(u) \geqslant C(x). \tag{17}$$

If we specialize to $s=z$, $g=u$ and a general relation between the cross-ratios of a function and its inverse follows:

$$C(u) \geqslant C(h). \tag{18}$$

Such a result is possible because of the condition that w shrinks the half-plane, i.e. $D \supseteq w(D)$.

It is interesting to consider the infinitesimal versions of some of our results. These invole the Schwarzian derivative

$$\{w, z\} \equiv \frac{w'''w' - \frac{3}{2}(w'')^2}{(w')^2}. \tag{19}$$

First consider (10) and let $x_2 = x_1 + \varepsilon$. Then to leading order one finds

$$\left(\frac{u_1 - u_2}{x_1 - x_2}\right)^2 - u_1' u_2' = -\frac{(u_1')^2}{6}\{u, x\}\varepsilon^2. \tag{20}$$

Thus

$$\{u, x\} \geqslant 0. \tag{21}$$

This result also follows from (17) on taking the appropriate limit [7].

The result (21) applies when u is the real part of a map w that shrinks the half-plane and x is such that u is on the real axis. Since the Schwarzian derivative is invariant under rotations and translations, the question arises whether (21) is also a sufficient condition for the continuation of u to be such a map. It is easy to see that this is not the case by consideration of the Schwarz–Christoffel formula, which maps the upper half-plane into a polygon. The Schwarzian derivative near the corners of the polygon is dominated by a divergent term. A straightforward calculation shows that the coefficient of this term is positive if the interior angle $\alpha < \pi$, but negative if $\alpha > \pi$. Thus (21) fails near an indentation, even though the overall map shrinks the half-plane. Of course, the boundary of a polygon near an indentation cannot be mapped onto the real axis without part of the polygon extending below it. Note also that (17) and (21) are saturated for bilinear maps, which include the transformation of the half-plane into a circle.

One may similarly reduce (16) to

$$\{h, g\} \geqslant 0. \tag{22}$$

Since Schwarzian derivatives follow the composition rule [7]

$$\{h, u\} = \left(\frac{\partial g}{\partial u}\right)^2 \{h, g\} + \{g, u\} \tag{23}$$

(22) leads immediately to

$$\{h, u\} \geqslant \{g, u\}. \tag{24}$$

Making use of the general relation $\{f, x\} = -\{x, f\}(\partial f/\partial x)^2$ [7] then gives

$$\frac{\{u, x\}}{u'^2} \leqslant \frac{\{q, e\}}{q'^2} \tag{25}$$

which involves the maps directly, instead of their inverses.

3. Applications

In this section we make a few applications of our results. We first consider some problems arising in the conformal field theory treatment of second-order phase transitions in two-dimensional systems, which provided the original motivation for this work. Then a new inequality for the Jacobian elliptic function sn u is derived. We also recall a connection wih previous research on scattering theory.

Consider a two-dimensional system confined to the upper half-plane at a conformally invariant second-order phase transition [8]. Now fix the thermodynamic state of the system boundary—the real axis, for the half-plane. Let the boundary state be A

for $x<x_1$, $x>x_2$ (with $x_1<x_2$) and B between x_1 and x_2. Then a (thermodynamic) domain boundary will be created. It (generally—see [8]) lies along a half-circle, with centre on the real axis, at $(x_2-x_1)/2$. Such a domain boundary is in fact created by boundary operators $\phi(x)$ [9, 10] at the points (x_1, x_2). Its extra free energy, i.e. the free energy difference between the system with a domain boundary and one with the single boundary state A (or B) along the entire system boundary, is given by

$$E=-\ln\langle\phi(x_1)\phi(x_2)\rangle \tag{26}$$

where the brackets denote a boundary operator correlation function. This function has a particularly simple form in the half-plane

$$\langle\phi(x_1)\phi(x_2)\rangle=\frac{1}{(x_2-x_1)^{2\Delta}} \tag{27}$$

where Δ is the critical dimension of the boundary operator, a pure number that depends on the universality class of the phase transition and the two boundary states involved. In the following, we assume (as is generally the case) that $\Delta\geqslant 0$. Thus the domain boundary energy in the upper half-plane is given by [8]

$$E_{1/2}=2\Delta\ln|x_1-x_2|. \tag{28}$$

It should be recognized that a domain boundary at criticality behaves rather differently from that in the usual situation. For example, its energy is not proportional to its length, and due to critical fluctuations, its position can only be defined on the average. See [8] and references therein for a more complete discussion.

Now in fact (26) is valid in any two-dimensional region, as long as the appropriate correlation function is used. But the functional form of the correlation function in any geometry attainable via a conformal map $w=w(z)$ of the half-plane follows immediately from the basic statement of conformal invariance of correlation functions [11]. Consider maps w and points (x_1, x_2) satisfying the conditions of section 2, so that the domain boundary runs between points on the real axis in the new geometry as well as in the half-plane. In these circumstances one finds immediately that

$$E_w=2\Delta\ln|(x_1-x_2)(u_1'u_2')^{1/2}|, \tag{29}$$

where x is to be taken a function of u, and the domain boundary runs between the points $u_1=u(x_1)$ and $u_2=u(x_2)$.

Now in general $E_{1/2}\neq E_w$ since both the geometries and the point locations differ. However we may pick $(x_1, x_2)=(u_1, u_2)$ in (29), so that only the geometry is changed. In the new geometry, there are fewer paths from u_1 to u_2, since (by hypothesis) w shrinks the half-plane. Therefore one expects the correlation fuction to be smaller, and by (26) $E_w\geqslant E_{1/2}$. It was this heuristic argument [12] that motivated the work presented here. Theorem 2 establishes this increase of the energy. Similarly, theorem 3 demonstrates that $E_w\geqslant E_s$. These results have many non-trivial consequences for domain boundary energies in restricted geometries, as explained in detail elsewhere [12].

As a simple example of the above, consider the map of the upper half-plane to a horizontal strip of width L defined by

$$w=\frac{L}{\pi}\ln(z). \tag{30}$$

If $x_1>x_2>0$, the conditions of the theorem are satisfied. Then one has the 'bubble' of

[8], with the points u_1 and u_2 on the lower edge of the strip. Here

$$E = 2\Delta \ln\left\{\frac{2L}{\pi}\sinh\left(\frac{\pi(u_2-u_1)}{2L}\right)\right\}$$

and (10) reduces to

$$y \leqslant \sinh(y), \tag{31}$$

where $y=\pi(u_2-u_1)/2L \geqslant 0$. For $x_2>0>x_1$, the two points are mapped to opposite sides of the strip (the 'wall' configuration [8]), the theorem does not apply, and the statement corresponding to (31) (with the sinh replaced by cosh) is incorrect.

Replacintg (30) by a projective (bilinear) transformation,

$$w = \frac{az+b}{cz+d} \tag{32}$$

with $ab-cd \neq 0$, one finds that the functional form of the energy (and the correlation function) is invariant. Here, as mentioned above, the inequality is saturated and the domain boundary energy E invariant (for the same separation of end points) [12]. In this case the half-plane can be mapped into a circle C. Then the mapped domain boundary follows the arc of a second circle, contained in C and intersecting it at right angles. Consider the straight line L between the end points of the domain boundary in C. The energy remains invariant because the reduction of paths caused by shrinking the half-plane to a circle on one side of L is exactly compensated by the addition of paths caused by the 'bowing out' of the segment of the real axis x into an arc of the circle C on the other side of L.

Equation (21) also has some implications for two-dimensional phase transitions. Consider a geometry attainable from the half-plane via a conformal map $w=w(z)$. Then the (thermodynamic average) stress tensor $\langle T\rangle$ in the w geometry is proportional to the Schwarzian derivative $\{w, z\}$, with a constant of proportionality that has a constant sign on the real z axis. On the other hand, the integral of $\langle T\rangle$ along the real axis contributes to a term in the free energy of the system. Whenever the central charge $c>0$ (which is the case for most transitions of physical interest), this contribution is of the same sign as the elastic energy of a system under tension. For a rectangle, for instance, there is a contribution of this sign along each edge, since the Schwarzian derivative is invariant under rotations and translations. The resulting overall term in the free energy provides a thermodynamic force for the elongation of the system [13]. It may be attributed to an attraction between the edges of the rectangle. Such elongation effects may have been observed experimentally [14]. For more details on the treatment of these terms, see [13, 15].

Equation (21) shows that such effects will be present quite generally. Conversely, the fact that $\{u, x\} \leqslant 0$ for an indentation, demonstrated above, implies that near such a feature this term is of the same sign as an elastic system under compression.

Consider now the Schwarz–Christoffel transformation, which maps the upper half-plane into a rectangle. This may be expressed via the Jacobian elliptic function [16]

$$z = \mathrm{sn}(w). \tag{33}$$

The inverse of (33) maps the points $x=\pm 1$, $\pm 1/k$ on the real axis onto the corners of the rectangle at $w=\pm K(k)$, $\pm K(k)+\mathrm{i}\,K'(k)$, respectively, where K and K' are complete elliptic integrals of the first kind with modulus $0<k<1$. Choose both points $x_1=0$ and $x_2=u<K(k)$. Making use of theorem 3, equation (30) and the 'bubble'

energy quoted above to compare the rectangle with a strip of width K' gives

$$\frac{\mathrm{sn}^2 u}{\mathrm{cn}\, u\, \mathrm{dn}\, u} \geqslant \left(\frac{2K'}{\pi} \sinh\left(\frac{\pi u}{2K'}\right)\right)^2 \tag{34}$$

where cn and dn are Jacobian elliptic functions. Each side of (34) being positive, one can invert and integrate between the limits $0<\delta<u$. Letting $\delta \to 0$, the leading (divergent) terms cancel and the remainders vanish, except the contributions of the upper limit of integration. On inversion and multiplication by -1, these give

$$\mathrm{sn}\, u \leqslant \frac{2K'}{\pi} \tanh\left(\frac{\pi u}{2K'}\right) \tag{35}$$

for $0<u<K$. To our knowledge, (35) is a new result. For $u \to 0$, the inequality saturates, the effects of the finite geometries becoming unimportant. For $k \to 1$, the rectangle becomes an infinite horizontal strip of width $K' = \pi/2$, $\mathrm{sn}\, u \to \tanh u$, and the inequality saturates, as it should since both geometries are the same. For $k \to 0$, $K' \to \infty$, $K \to \pi/2$, $\mathrm{sn}\, u \to \sin u$, and the inequality reduces to $\sin u \leqslant u$.

Equation (35) may also be obtained directly from (16) with $g = \mathrm{e}^{(\pi/K')u}$, $h = \mathrm{sn}\, u$, setting $u_1 = -u$, $u_2 = -\varepsilon$, $u_3 = \varepsilon$, $u_4 = u$; $\varepsilon, u > 0$ and taking the limit $\varepsilon \to 0$. Further results for the Jacobian elliptic functions obtained along these lines will be published elsewhere [17].

Finally we mention that use has been made of monotone matrix functions in the theory of scattering [18, 19]. These papers consider the $\boldsymbol{R}$ matrix, which connects matrix elements of the wavefunction with matrix elements of its normal derivative on a given surface. It is argued that the kernel formed from this matrix, considered as a function of the enery E, is positive definite, and hence R is the restriction to the real axis of a conformal map of the half-plane into itself. The consequences of this are used to established a causality condition and the existence of a particular expansion for $\boldsymbol{R}$.

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